Utrecht University

EINDHOVEN UNIVERSITY OF TECHNOLOGY

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Introducing formalization of mathematics with Tutte's theorem

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Outline

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About me

- Pim Otte
- PhD candidate at Utrecht University & Technical University of Eindhoven
- Topic “Type Theory for Education”
- Webpage: `https://pim.otte.dev`

Topic

- Formalization of mathematics: What and Why?
- Demo: Formalizing Tutte's theorem

Outline

Formalization

- What: Encoding mathematics in a formal system

Formalization

- What: Encoding mathematics in a formal system
- Why: To teach mathematics to computers

Formalization

- What: Encoding mathematics in a formal system
- Why: To teach mathematics to computers
- Why: To create proof assistants (Interactive Theorem Provers)

Why do we want proof assistants?

- Verification

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- To build tools to help mathematicians

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- Verification
- To build tools to help mathematicians
- To scale mathematical collaboration

Verification

- Computer assisted proofs: The four colour theorem [G$^+$08]

Verification

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- Mathematical doubt: The Liquid Tensor Experiment [Buz22] [Com21]

Tools for mathematicians

- Proof *assistant*

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  - Interactive theorem proving

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    - Interactive theorem proving
    - Theorem search

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Tools for mathematicians

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  - Interactive theorem proving
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- Teaching tools

Scaling mathematical collaboration

- What's the biggest mathematical collaboration in the room?

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- Liquid Tensor Experiment (29 contributors)

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- Equational Theories (56 contributors)

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- Fermat's Last Theorem (ongoing) (44 contributors)
- Equational Theories (56 contributors)
- Mathlib (413 contributors)

Bonus: Interaction with Generative AI

- Problem: Generative AI is hit or miss

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- Examples: AlphaProof [AA24], Deepseek Prover [XGS$^+$24]

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Curry-Howard correspondence

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- A function $f : A \to B$ turns a proof of A into a proof of B.
- Similarly: for $P : \mathbb{R} \to \mathrm{Prop}$, $\forall x : \mathbb{R}, P(x)$ corresponds to a function that provides a proof of $P(x)$ on an input x.

Curry-Howard correspondence

```
theorem example1 {A : Prop} : A → A := fun (a : A) => a

theorem example2 {P : ℕ → Prop} :
    (∀ (n : ℕ), P n) → P 37 := fun Pforall => Pforall 37
```

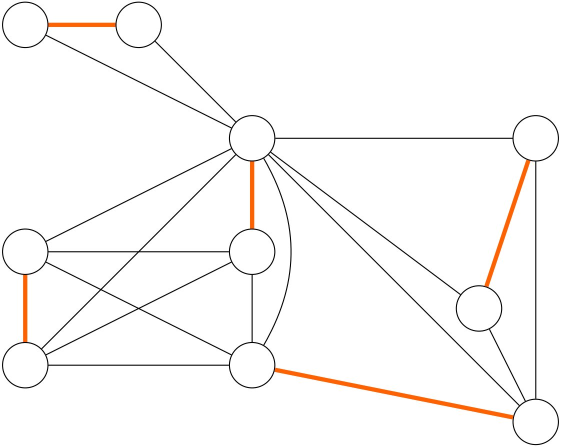
Tactics Demo

Tactics Demo

```lean
import Mathlib

theorem rearranging (x y z : ℝ) :
    (5*x + y) * z + (3*z)*y = 5*x*z + 4*y*z := by
  ring

theorem integer_inequalities (n m : ℤ)
    (h : n < 5 * m) (h2 : 5 + n > -m) : m > -1 := by
  omega

theorem linear_inequalities (a b : ℝ) (h : 5*a + b < 10) (h2
    : 9*a - 5*b > -20) :  b < 6 := by
  linarith
```

Outline

Perfect Matching



Perfect Matching

Definition (Perfect matching).

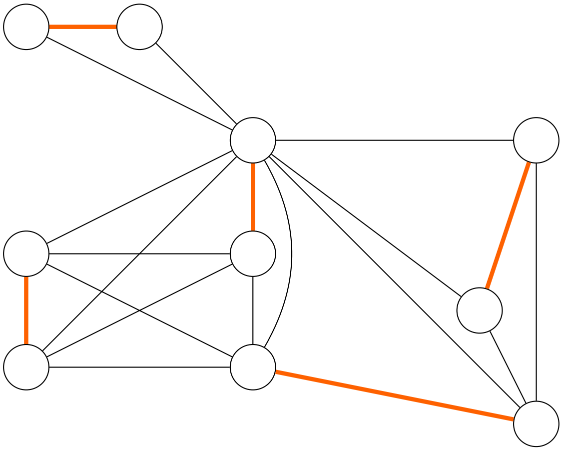
A perfect matching of G is a subgraph M, such that every vertex in G is connected in M to exactly one other vertex.

Tutte's theorem

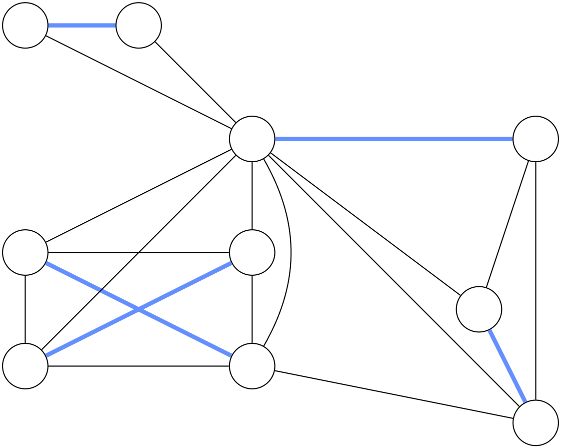
Theorem (Tutte, 1947).

A graph G has a perfect matching if and only if for any subset $U \subseteq V$ the graph $G - U$ has at most $|U|$ components of odd size.

Alternating cycles



Alternating cycles



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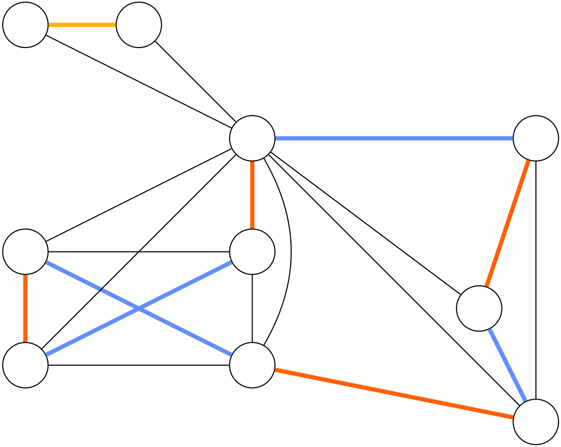

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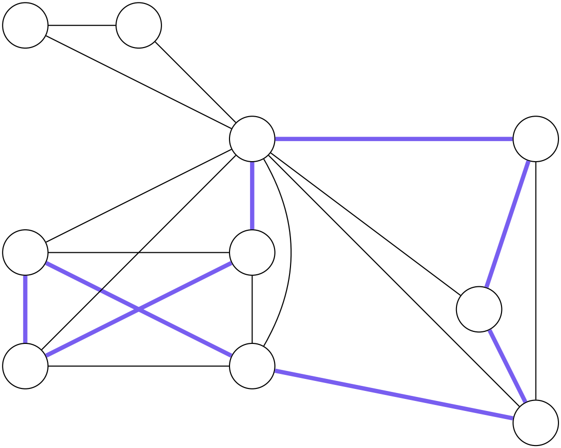

Alternating graph

Definition (Alternating).

A graph G is alternating with respect to some other graph G' if exactly every other edge in G is in G'.

Symmetric difference

Definition (Symmetric difference).

The symmetric difference of G and G' (denoted by $G \Delta G'$) is the graph consisting of edges that are in exactly one of G and G'.

Symmetric difference of perfect matchings is alternating

Lemma.

Let M, M' be perfect matchings of a graph G, then the symmetric difference $M \Delta M'$ is alternating with respect to M.

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Proof.

It suffices to show that for any vertex v in $M \Delta M'$ and two edges including v, exactly of these edges is in M. By definition of $M \Delta M'$, both of these edges must be in M or M'. Since M and M' are perfect matchings, each edge must be in exactly one of the two. So in particular, exactly one of these edges is in M. □

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Introduction

Motivation

Concepts

Tutte’s theorem

Demo

Conclusion

Demo

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```
lemma
  Subgraph.IsPerfectMatching.isAlternating_symmDiff_left
    {M' : Subgraph G'} (hM : M.IsPerfectMatching)
    (hM' : M'.IsPerfectMatching) :
    (M.spanningCoe △ M'.spanningCoe).IsAlternating
  M.spanningCoe := by
  sorry
```

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  - Lean, Rocq, Isabelle, Agda

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  - Do/join/supervise a project

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- Teaching computers mathematics: verification, tools and collaboration
- How to get started?
  - Lean, Rocq, Isabelle, Agda
  - Natural number game: https://adam.math.hhu.de/
  - Do/join/supervise a project
  - Utrecht Summer School

References

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shaping tomorrow

Thanks to Yaël Dillies, Bhavik Mehta, Kyle Miller and other mathlib reviewers.

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